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J. Math. Anal. Appl. 339 (2008) 553–565

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Precise rates in complete moment convergence for ρ-mixing sequences ☆

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Received 27 September 2006

Available online 19 July 2007

Submitted by M. Peligrad

Abstract

Let $X_1, X_2, \ldots$ be a strictly stationary sequence of ρ-mixing random variables with mean zeros and positive, finite variances, set $S_n = X_1 + \cdots + X_n$. Suppose that $\lim_{n\to\infty} ES_n^2/n = \sigma^2 > 0$, $\sum_{n=1}^{\infty} \rho^{2/q}(2^n) < \infty$, where $q > 2\delta + 2$. We prove that, if $EX_1^2(\log^+ |X_1|)^\delta < \infty$ for any $0 < \delta \leqslant 1$, then

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} ES_n^2 I\left(|S_n| \geqslant \epsilon\sigma\sqrt{n\log n}\right) = \frac{E|N|^{2\delta+2}}{\delta},$$

where N is the standard normal random variable.

Keywords: Precise rates; Complete moment convergence; ρ-Mixing; Mixing rates

1. Introduction and the main results

Suppose that $\{X_n: n \geqslant 1\}$ is a sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$, set $\mathcal{F}_n^- = \sigma(X_i: 1 \leqslant i \leqslant n)$, $\mathcal{F}_n^+ = \sigma(X_i: i \geqslant n)$,

$$\rho(n) := \sup_{k\geqslant 1} \sup_{X\in L_2(\mathcal{F}_k^-)} \sup_{Y\in L_2(\mathcal{F}_{k+n}^+)} \frac{|EXY - EXEY|}{\sqrt{E(X-EX)^2E(Y-EY)^2}}, \tag{1.1}$$

the sequence $\{X_n: n \geqslant 1\}$ is said to be ρ-mixing, if $\rho(n) \to 0$ as $n \to \infty$.

This definition was introduced by Kolmogorov and Rozanov [7], and the limiting behaviors of ρ-mixing sequences have received more and more attention recently. Under appropriate mixing rates, lots of limit theorems have been obtained. The central limit theorem (CLT) was proved by Ibragimov [6], Bradley [1,2], Dehling, Denker and Philipp [4], Peligrad [12–14]. The functional central limit theorem (FCLT) is due to Shao [18]. Further results are probability in-

☆ This work is supported by the Natural Science Foundation of Department of Education of Zhejiang Province (Grant No. 20060237).
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.06.070

equalities (see, e.g., [20,22]), invariance principle (cf. [10,15,16,18,21]), the complete convergence (see, e.g., [11,19]), and so forth.

Note that in the above-mentioned limit theorems, the precise rates in the complete moment convergence for ρ-mixing sequences are little known. The aim of the present work is to investigate this asymptotic behavior, of course, our results have the connection with complete moment convergence. It is well known that a lot of beautiful results have been established for independent random variables. Let $\{X_n\colon n \geqslant 1\}$ be i.i.d. (independent and identically distributed) nondegenerate random variables. Chow [3] obtained a result as follows.

Theorem A. *Suppose that $EX_1 = 0$ and $E(|X_1|^r + |X_1|\log(1+|X_1|)) < \infty$, for $r > p$, $1 \leqslant p < 2$. Then*

$$\sum_{n=1}^{\infty} n^{r/p-2-1/p} E\big(|S_n| - \epsilon n^{1/p}\big)_+ < \infty, \quad \epsilon > 0. \tag{1.2}$$

A natural question is that how larger the convergence rate of (1.2) is as $\epsilon \downarrow 0$? As we know this is an interesting question, one of the answers to moving-average process reads as follows:

$$\lim_{\epsilon\downarrow 0} \epsilon^{2(r-p)/(2-p)-1} \sum_{n=1}^{\infty} n^{r/p-2-1/p} E\big(|S_n| - \epsilon n^{1/p}\big)_+ = \frac{p(2-p)}{(r-p)(2r-p-2)} E|N|^{2(r-p)/(2-p)}, \tag{1.3}$$

where N is the standard normal random variable. For more details, refer to [9, Theorem 1.1]. The following theorem for i.i.d. random variables was recently proved by Liu and Lin [8].

Theorem B. *Suppose that*

$$EX_1 = 0, \qquad EX_1^2 = \sigma^2 \quad \text{and} \quad EX_1^2\big(\log^+|X_1|\big)^{\alpha} < \infty, \tag{1.4}$$

for $0 < \alpha \leqslant 1$. Then

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\alpha} \sum_{n=2}^{\infty} \frac{(\log n)^{\alpha-1}}{n^2} ES_n^2 I\big(|S_n| \geqslant \epsilon\sqrt{n\log n}\big) = \frac{\sigma^{2\alpha+2}}{\alpha} E|N|^{2\alpha+2}. \tag{1.5}$$

Conversely, if (1.5) *is true, then* (1.4) *holds.*

Should (1.5) be true for ρ-mixing sequences under appropriate conditions? Furthermore, if S_n is replaced by $\max_{1\leqslant k\leqslant n}|S_k|$, what can be said about the precise asymptotics? The present work gives the answers. From now, it is very convenient to adopt the following notations throughout the paper: let $X_1, X_2, \ldots$ be strictly stationary ρ-mixing sequences with $EX_1 = 0$ and $EX_1^2 < \infty, \sigma^2 = EX_1^2 + 2\sum_{n=2}^{\infty} EX_1X_n > 0$, and set $S_n = X_1 + \cdots + X_n$, $M_n = \max_{1\leqslant k\leqslant n}|S_k|$, $\log x = \log_e(x \vee e)$, $[z]$ denotes the largest integer which is not lager than z, the letter C with subscripts denotes some finite and positive universal constants not important in our investigations, it may take different values in each appearance. The organization of the paper is as follows. We first introduce our main results, after which the proofs are exposed in Sections 2 and 3.

Theorem 1.1. *Suppose that $\lim_{n\to\infty} ES_n^2/n = \sigma^2 > 0$, and $\sum_{n=1}^{\infty}\rho^{2/q}(2^n) < \infty$, where $q > 2\delta + 2$, $EX_1^2(\log|X_1|)^{\delta} < \infty$ for any $0 < \delta \leqslant 1$. Then*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} ES_n^2 I\big(|S_n| \geqslant \epsilon\sigma\sqrt{n\log n}\big) = \frac{E|N|^{2\delta+2}}{\delta}. \tag{1.6}$$

Theorem 1.2. *Under the conditions of Theorem* 1.1, *for any $0 < \delta \leqslant 1$,*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} EM_n^2 I\big(M_n \geqslant \epsilon\sigma\sqrt{n\log n}\big) = \frac{2E|N|^{2\delta+2}}{\delta} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2\delta+2}}. \tag{1.7}$$

If the mixing rate $\sum_{n=1}^{\infty}\rho^{2/q}(2^n)<\infty$ is replaced by $\sum_{n=1}^{\infty}\rho(n)<\infty$, one can immediately obtain the following result.

Theorem 1.3. *Suppose that* $\lim_{n\to\infty} ES_n^2/n=\sigma^2>0$, *and* $\sum_{n=1}^{\infty}\rho(n)<\infty$. *Then, for any* $0<\delta\leqslant 1$,

$$EX_1=0 \quad \textit{and} \quad EX_1^2\big(\log|X_1|\big)^{\delta}<\infty \tag{1.8}$$

if and only if

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}EM_n^2I\big(M_n\geqslant\epsilon\sigma\sqrt{n\log n}\big)=\frac{2E|N|^{2\delta+2}}{\delta}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2\delta+2}}. \tag{1.9}$$

Without loss of generality, in the sequel, we will suppose that $\sigma^2=1$.

2. Proof of Theorem 1.1

We begin this section by introducing two lemmas, which are helpful in proving the theorem.

Lemma 2.1. *(See Ibragimov [6].) Assume that* $\{X_n\colon n\geqslant 1\}$ *is a sequence of strictly stationary* ρ*-mixing random variables with* $EX_1=0$ *and* $EX_1^2<\infty, \sigma_n^2=ES_n^2\to\infty$ *as* $n\to\infty$ *and* $\sum_{n=1}^{\infty}\rho(2^n)<\infty$. *Then*

$$S_n/\sigma_n\xrightarrow{\mathscr{D}}N(0,1)\quad \textit{as } n\to\infty. \tag{2.1}$$

Lemma 2.2. *(See Shao [22].) Let* $\{X_n\colon n\geqslant 1\}$ *be a sequence of* ρ*-mixing random variables. Assume that* $EX_n=0$, *put* $S_k(n)=\sum_{i=k+1}^{k+n}X_i$, $k\geqslant 0$. *Then, for any* $q\geqslant 2$, *there exists* $K=K(q,\rho(\cdot))$ *depending only on* q *and* $\rho(\cdot)$ *such that*

$$\begin{aligned}P\Big(\max_{1\leqslant j\leqslant n}\big|S_k(j)\big|\geqslant x\Big)\leqslant{}&\sum_{i=k+1}^{k+n}P\big(|X_i|\geqslant y\big)+Kx^{-q}n^{q/2}\exp\Bigg(K\sum_{i=1}^{[\log n]}\rho\big(2^i\big)\Bigg)\max_{k\leqslant i\leqslant k+n}\big\|X_iI\big(|X_i|\leqslant y\big)\big\|_2^q\\&+Kx^{-q}n\exp\Bigg(K\sum_{i=1}^{[\log n]}\rho^{2/q}\big(2^i\big)\Bigg)\max_{k\leqslant i\leqslant k+n}E|X_i|^qI\big(|X_i|\leqslant y\big),\end{aligned}$$

for any $x>0$ *and* $y>0$ *with* $2n\max_{k\leqslant i\leqslant k+n}E|X_i|I(|X_i|\geqslant y)\leqslant x$.

In fact, one can easily get

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}ES_n^2I\big(|S_n|\geqslant\epsilon\sqrt{n\log n}\big)&=\epsilon^2\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\big(|S_n|\geqslant\epsilon\sqrt{n\log n}\big)\\&\quad+\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(|S_n|\geqslant x\big)\,dx\\&=:I_1+I_2.\end{aligned}$$

Consequently, to verify (1.6), we only need to study I_1 and I_2, respectively. The rest of this section is to present several propositions which are needed in the proof.

Proposition 2.1. *Suppose that* N *is a standard normal random variable. Then, for any* $0<\delta\leqslant 1$,

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta+2}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\big(|N|\geqslant\epsilon\sqrt{\log n}\big)=\frac{E|N|^{2\delta+2}}{\delta+1}. \tag{2.2}$$

Proof. For the proof see [5, Theorem 1.3]. □

Proposition 2.2. *Suppose that* $\lim_{n\to\infty} ES_n^2/n = \sigma^2 > 0$, *and* $\sum_{n=1}^{\infty} \rho^{2/q}(2^n) < \infty$, *where* $q > 2\delta + 2$, $EX_1^2(\log|X_1|)^\delta < \infty$ *for any* $0 < \delta \leqslant 1$. *Then*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta+2} \sum_{n=1}^{\infty} \frac{(\log n)^\delta}{n} \left| P\left(|S_n| \geqslant \epsilon\sqrt{n\log n}\right) - P\left(|N| \geqslant \epsilon\sqrt{\log n}\right)\right| = 0. \tag{2.3}$$

Proof. Using the standard method, set $H(\epsilon) = [\exp(M/\epsilon^2)]$, where $M > 4$, $0 < \epsilon < 1/4$. It is easy to get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(\log n)^\delta}{n} \left| P\left(|S_n| \geqslant \epsilon\sqrt{n\log n}\right) - P\left(|N| \geqslant \epsilon\sqrt{\log n}\right)\right| \\
&\quad = \sum_{n\leqslant H(\epsilon)} \frac{(\log n)^\delta}{n} \left| P\left(|S_n| \geqslant \epsilon\sqrt{n\log n}\right) - P\left(|N| \geqslant \epsilon\sqrt{\log n}\right)\right| \\
&\qquad + \sum_{n> H(\epsilon)} \frac{(\log n)^\delta}{n} \left| P\left(|S_n| \geqslant \epsilon\sqrt{n\log n}\right) - P\left(|N| \geqslant \epsilon\sqrt{\log n}\right)\right| \\
&\quad =: \Sigma_1 + \Sigma_2.
\end{aligned}$$

Write $\Delta_n = \sup_x |P(|S_n| \geqslant x\sqrt{n}) - P(|N| \geqslant x)|$, note that $P(|N| \geqslant x)$ is a continuous function for $x \geqslant 0$, and this combined with Lemma 2.1 yields $\lim_{n\to\infty} \Delta_n = 0$ for any $x \geqslant 0$. Then, applying Toeplitz's lemma (see, e.g., [17, Lemma 6.10]), it follows that

$$\begin{aligned}
\epsilon^{2\delta+2}\Sigma_1 &\leqslant \epsilon^{2\delta+2} \sum_{n\leqslant H(\epsilon)} \frac{(\log n)^\delta}{n} \Delta_n \\
&= \epsilon^{2\delta+2}\left(\log H(\epsilon)\right)^{\delta+1} \frac{1}{\log(H(\epsilon))^{\delta+1}} \sum_{n\leqslant H(\epsilon)} \frac{(\log n)^\delta}{n} \Delta_n \to 0 \quad \text{as } \epsilon \downarrow 0.
\end{aligned} \tag{2.4}$$

Obviously, we have, for the second part Σ_2,

$$\begin{aligned}
\Sigma_2 &\leqslant \sum_{n>H(\epsilon)} \frac{(\log n)^\delta}{n} P\left(|N| \geqslant \epsilon\sqrt{\log n}\right) + \sum_{n>H(\epsilon)} \frac{(\log n)^\delta}{n} P\left(|S_n| \geqslant \epsilon\sqrt{n\log n}\right) \\
&=: \Sigma_3 + \Sigma_4.
\end{aligned}$$

Notice that $H(\epsilon) - 1 \geqslant \sqrt{H(\epsilon)}$ for $M > 4$ and $0 < \epsilon < 1/4$, an easy calculation leads to

$$\begin{aligned}
\epsilon^{2\delta+2}\Sigma_3 &\leqslant \epsilon^{2\delta+2} \sum_{n>H(\epsilon)} \frac{(\log n)^\delta}{n} P\left(|N| \geqslant \epsilon\sqrt{\log n}\right) \\
&\leqslant C \int_{\sqrt{M/4}}^{\infty} y^{2\delta+1} P\left(|N| > y\right) dy \to 0 \quad \text{as } M \to \infty,
\end{aligned} \tag{2.5}$$

uniformly with respect to $0 < \epsilon < 1/4$. For Σ_4, taking $x = \epsilon\sqrt{n\log n}$, $y = 2\epsilon\sqrt{n\log n}$ in Lemma 2.2. Note that $n > H(\epsilon)$ if and only if $\log n > M/\epsilon^2$, it turns out that

$$\frac{2n \max_{1\leqslant i\leqslant n} E|X_i| I(|X_i| \geqslant 2\epsilon\sqrt{n\log n})}{\epsilon\sqrt{n\log n}} \leqslant C \frac{E|X_i|^2 I(|X_i| \geqslant 2\epsilon\sqrt{n\log n})}{\epsilon^2 \log n} \leqslant C/M \to 0 \quad \text{as } M \to \infty. \tag{2.6}$$

Furthermore, applying Lemma 2.2, we have

$$P\left(\max_{1\leqslant j\leqslant n} |S_j| \geqslant x\right) \leqslant nP\left(|X_1| \geqslant y\right) + Kx^{-q} n^{q/2} \exp\left(K \sum_{i=1}^{[\log n]} \rho\left(2^i\right)\right) \left(EX_1^2 I\left(|X_1| \leqslant y\right)\right)^{q/2}$$

$$
\begin{aligned}
&+ Kx^{-q} n \exp\left(K \sum_{i=1}^{[\log n]} \rho^{2/q}\left(2^i\right)\right) E|X_1|^q I\left(|X_1| \leqslant y\right) \\
&=: \Sigma_5 + \Sigma_6 + \Sigma_7.
\end{aligned}
$$

Recalling the moment condition, which in turn implies

$$
\begin{aligned}
\sum_{n>H(\epsilon)} n^{-1}(\log n)^\delta \Sigma_5 &= \sum_{n>H(\epsilon)} (\log n)^\delta P\left(|X_1| \geqslant 2\epsilon\sqrt{n\log n}\right) \\
&\leqslant C \sum_{n>H(\epsilon)} (\log n)^\delta \sum_{k=n}^{\infty} P\left(2\epsilon\sqrt{k\log k} \leqslant |X_1| < 2\epsilon\sqrt{(k+1)\log(k+1)}\right) \\
&\leqslant C \sum_{k>H(\epsilon)} P\left(2\epsilon\sqrt{k\log k} \leqslant |X_1| < 2\epsilon\sqrt{(k+1)\log(k+1)}\right) \sum_{n=1}^{k} (\log n)^\delta \\
&\leqslant C \sum_{k>H(\epsilon)} k(\log k)^\delta P\left(k \leqslant X_1^2/4M < k+1\right) \\
&\leqslant CEX_1^2\left(\log|X_1|\right)^\delta I\left(|X_1| \geqslant 2\epsilon\sqrt{H(\epsilon)\log H(\epsilon)}\right) < \infty. \qquad (2.7)
\end{aligned}
$$

Since $q > 2\delta + 2$, it suffices to show that

$$
\begin{aligned}
\sum_{n>H(\epsilon)} n^{-1}(\log n)^\delta \Sigma_6 &\leqslant C \sum_{n>H(\epsilon)} \frac{(\log n)^\delta}{n} (\epsilon\sqrt{n\log n})^{-q} n^{q/2} \\
&\leqslant C\epsilon^{-q} \sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta-q/2} \leqslant C\epsilon^{-q} \int_{H(\epsilon)-1}^{\infty} x^{-1}(\log x)^{\delta-q/2}\,dx \\
&\leqslant C\epsilon^{-q} \int_{\sqrt{H(\epsilon)}}^{\infty} x^{-1}(\log x)^{\delta-q/2}\,dx \leqslant C\epsilon^{-2(\delta+1)} M^{\delta-q/2+1}, \qquad (2.8)
\end{aligned}
$$

consequently, we have $\lim_{M\to\infty} \epsilon^{2\delta+2} \sum_{n>H(\epsilon)} n^{-1}(\log n)^\delta \Sigma_6 = 0$. Finally, focusing attention on Σ_7, It turns out that

$$
\begin{aligned}
&\sum_{n>H(\epsilon)} n^{-1}(\log n)^\delta \Sigma_7 \\
&\leqslant C\epsilon^{-q} \sum_{n>H(\epsilon)} n^{-q/2}(\log n)^{\delta-q/2} E|X_1|^q I\left(|X_1| \leqslant 2\epsilon\sqrt{n\log n}\right) \\
&\leqslant C\epsilon^{-q} \sum_{n>H(\epsilon)} n^{-q/2}(\log n)^{\delta-q/2} \sum_{1\leqslant j\leqslant n} E|X_1|^q I\left(2\epsilon\sqrt{j\log j} \leqslant |X_1| \leqslant 2\epsilon\sqrt{(j+1)\log(j+1)}\right) \\
&\leqslant C\epsilon^{-q} \sum_{j>H(\epsilon)} j^{-q/2+1}(\log j)^{\delta-q/2} E|X_1|^q I\left(2\epsilon\sqrt{j\log j} \leqslant |X_1| \leqslant 2\epsilon\sqrt{(j+1)\log(j+1)}\right) \\
&\leqslant C\epsilon^{-2} \sum_{j>H(\epsilon)} (\log j)^{\delta-1} E|X_1|^2 I\left(2\epsilon\sqrt{j\log j} \leqslant |X_1| \leqslant 2\epsilon\sqrt{(j+1)\log(j+1)}\right) \\
&\leqslant C\epsilon^{-2} E|X_1|^2 I\left(|X_1| \geqslant 2\epsilon\sqrt{H(\epsilon)\log H(\epsilon)}\right). \qquad (2.9)
\end{aligned}
$$

Notice, in particular, that $n > H(\epsilon)$, it follows that $\epsilon^{2\delta} E|X_1|^2 I(|X_1| \geqslant 2\epsilon\sqrt{H(\epsilon)\log H(\epsilon)}) \to 0$ as $\epsilon \downarrow 0$ and $M \to \infty$. Putting (2.4), (2.5), (2.7)–(2.9) together, we obtain (2.3). □

Proposition 2.3. *Suppose that N is a standard normal random variable. Then, for any* $0 < \delta \leqslant 1$,

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(|N|\geqslant x/\sqrt{n}\big)=\frac{E|N|^{2\delta+2}}{\delta(\delta+1)}. \tag{2.10}$$

Proof. Refer to Liu and Lin [8]. □

Proposition 2.4. *Under the conditions of Theorem* 1.1*,*

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n\leqslant H(\epsilon)}\frac{(\log n)^{\delta-1}}{n^2}\left|\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(|S_n|\geqslant x\big)\,dx-\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(|N|\geqslant x/\sqrt{n}\big)\,dx\right|=0. \tag{2.11}$$

Proof. Set $H(\epsilon)=[\exp(M/\epsilon^2)]$ and denote $\Delta_n=\sup_x|P(|S_n|\geqslant\sqrt{n}x)-P(|N|\geqslant x)|$. Then, assume that $x=(y+\epsilon)\sqrt{n\log n}$, by integral formula and transformation, it is enough to show that

$$\begin{aligned}
&\sum_{n\leqslant H(\epsilon)}n^{-2}(\log n)^{\delta-1}\left|\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(|S_n|\geqslant x\big)\,dx-\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\big(|N|\geqslant x/\sqrt{n}\big)\,dx\right|\\
&\leqslant C\sum_{n\leqslant H(\epsilon)}n^{-1}(\log n)^{\delta}\int_0^{\infty}2(y+\epsilon)\big|P\big(|S_n|\geqslant(y+\epsilon)\sqrt{n\log n}\big)-P\big(|N|\geqslant(y+\epsilon)\sqrt{\log n}\big)\big|\,dy\\
&\leqslant C\sum_{n\leqslant H(\epsilon)}n^{-1}(\log n)^{\delta}\left\{\int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty}2(y+\epsilon)P\big(|N|\geqslant(y+\epsilon)\sqrt{\log n}\big)\,dy\right.\\
&\quad+\int_0^{1/\sqrt{\log n}\Delta_n^{1/4}}2(y+\epsilon)\big|P\big(|S_n|\geqslant(y+\epsilon)\sqrt{n\log n}\big)-P\big(|N|\geqslant(y+\epsilon)\sqrt{\log n}\big)\big|\,dy\\
&\quad\left.+\int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty}2(y+\epsilon)P\big(|S_n|\geqslant(y+\epsilon)\sqrt{n\log n}\big)\,dy\right\}\\
&=:C\sum_{n\leqslant H(\epsilon)}\frac{(\log n)^{\delta}}{n}(\Lambda_1+\Lambda_2+\Lambda_3).
\end{aligned}$$

The estimates of Λ_1 and Λ_2 are similar to those of Proposition 5.2 in [8], so we omit them. It remains to estimate Λ_3, taking $x=(y+\epsilon)\sqrt{n\log n}$, $y=2(y+\epsilon)\sqrt{n\log n}$ in Lemma 2.2, which yields

$$\begin{aligned}
\sum_{n\leqslant H(\epsilon)}n^{-1}(\log n)^{\delta}\Lambda_3&\leqslant C\sum_{n\leqslant H(\epsilon)}(\log n)^{\delta}\int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty}2(y+\epsilon)P\big(|X_1|\geqslant 2(y+\epsilon)\sqrt{n\log n}\big)\,dy\\
&\quad+C\sum_{n\leqslant H(\epsilon)}n^{-1}(\log n)^{\delta-q/2}\int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty}(y+\epsilon)^{1-q}\exp\left(\sum_{i=1}^{[\log n]}\rho\big(2^i\big)\right)dy\\
&\quad+C\sum_{n\leqslant H(\epsilon)}n^{-q/2}(\log n)^{\delta-q/2}\int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty}(y+\epsilon)^{1-q}\exp\left(\sum_{i=1}^{[\log n]}\rho^{2/q}\big(2^i\big)\right)\\
&\quad\times E|X_1|^qI\big(|X_1|\leqslant 2(y+\epsilon)\sqrt{n\log n}\big)\,dy=:\Lambda_4+\Lambda_5+\Lambda_6.
\end{aligned}$$

Note that $(\log H(\epsilon))^\delta = M^\delta/\epsilon^{2\delta}$ and $EX_1^2 I(|X_1| \geqslant \sqrt{n}) \to 0$ as $n \to \infty$. By Toeplitz's lemma, it follows that

$$\begin{aligned}
\epsilon^{2\delta}\Lambda_4 &\leqslant \epsilon^{2\delta} \sum_{n\leqslant H(\epsilon)} (\log n)^\delta E\left(\int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty} 2(y+\epsilon) I\big(|X_1|/2\sqrt{n\log n} \geqslant y+\epsilon\big)\right) dy \\
&\leqslant \epsilon^{2\delta} \sum_{n\leqslant H(\epsilon)} \frac{EX_1^2 I(|X_1| \geqslant \sqrt{n})(\log n)^{\delta-1}}{n} \\
&\leqslant M^\delta \big(1/\big(\log H(\epsilon)\big)^\delta\big) \sum_{n\leqslant H(\epsilon)} \frac{EX_1^2 I(|X_1| \geqslant \sqrt{n})(\log n)^{\delta-1}}{n} \to 0 \quad \text{as } \epsilon \downarrow 0. \qquad (2.12)
\end{aligned}$$

Observe that $\Delta_n^{(q-2)/4} \to 0$ as $n \to \infty$. using Toeplitz's lemma again, we have

$$\begin{aligned}
\epsilon^{2\delta}\Lambda_5 &\leqslant \epsilon^{2\delta} \sum_{n\leqslant H(\epsilon)} n^{-1}(\log n)^{\delta-q/2} \int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty} (y+\epsilon)^{1-q}\, dy \\
&\leqslant \epsilon^{2\delta} \sum_{n\leqslant H(\epsilon)} \frac{\Delta_n^{(q-2)/4}(\log n)^{\delta-1}}{n} \\
&\leqslant M^\delta \big(1/\big(\log H(\epsilon)\big)^\delta\big) \sum_{n\leqslant H(\epsilon)} \frac{\Delta_n^{(q-2)/4}(\log n)^{\delta-1}}{n} \to 0 \quad \text{as } \epsilon \downarrow 0. \qquad (2.13)
\end{aligned}$$

Finally, let us estimate Λ_6, recalling $q > 2\delta + 2$. It turns out that

$$\begin{aligned}
&\sum_{n\leqslant H(\epsilon)} n^{-q/2}(\log n)^{\delta-q/2} \int_{1/\sqrt{\log n}\Delta_n^{1/4}}^{\infty} (y+\epsilon)^{1-q} E|X_1|^q I\big(|X_1| \leqslant 2(y+\epsilon)\sqrt{n\log n}\big)\, dy \\
&\leqslant C \sum_{n\leqslant H(\epsilon)} n^{-q/2}(\log n)^{\delta-q/2} E|X_1|^q \int_0^\infty (y+\epsilon)^{1-q} I\big(|X_1| \leqslant \epsilon\sqrt{n\log n}\big)\, dy \\
&\quad + C \sum_{n\leqslant H(\epsilon)} n^{-q/2}(\log n)^{\delta-q/2} E|X_1|^q \int_0^\infty (y+\epsilon)^{1-q} I\big(\epsilon\sqrt{n\log n} < |X_1| \leqslant (y+\epsilon)\sqrt{n\log n}\big)\, dy \\
&=: \Lambda_7 + \Lambda_8.
\end{aligned}$$

Several lines of elementary calculation yield

$$\begin{aligned}
\Lambda_7 &\leqslant C\epsilon^{2-q} \sum_{n\leqslant H(\epsilon)} n^{-q/2}(\log n)^{-q/2+\delta} \sum_{i=1}^{n} E|X_1|^q I\big(\epsilon\sqrt{i\log i} \leqslant |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \\
&= C\epsilon^{2-q} \sum_{i\leqslant H(\epsilon)} E|X_1|^q I\big(\epsilon\sqrt{i\log i} \leqslant |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \sum_{n=i}^{\infty} n^{-q/2}(\log n)^{-q/2+\delta} \\
&\leqslant C \sum_{i\leqslant H(\epsilon)} (\log i)^{-1} E|X_1|^2 \big(\log|X_1|\big)^\delta I\big(\epsilon\sqrt{i\log i} \leqslant |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \\
&\leqslant C \sum_{i\leqslant H(\epsilon)} E|X_1|^2 \big(\log|X_1|\big)^\delta I\big(\epsilon\sqrt{i\log i} \leqslant |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \\
&\leqslant CE|X_1|^2\big(\log|X_1|\big)^\delta < \infty. \qquad (2.14)
\end{aligned}$$

In view of the moment condition, it follows that

$$\begin{aligned}
\Lambda_8 &\leqslant C \sum_{n\leqslant H(\epsilon)} n^{-q/2}(\log n)^{-q/2+\delta} E|X_1|^q I\big(|X_1| > \epsilon\sqrt{n\log n}\big) \int_0^\infty (x+\epsilon)^{1-q} I\big(|X_1| \leqslant 2(x+\epsilon)\sqrt{n\log n}\big)\,dx \\
&\leqslant C \sum_{n\leqslant H(\epsilon)} n^{-q/2}(\log n)^{-q/2+\delta} E|X_1|^q I\big(|X_1| > \epsilon\sqrt{n\log n}\big) \int_{\frac{|X_1|}{2\sqrt{n\log n}}}^\infty (x+\epsilon)^{1-q}\,dx \\
&\leqslant C \sum_{n\leqslant H(\epsilon)} n^{-1}(\log n)^{-1+\delta} \sum_{i=n}^\infty E|X_1|^2 I\big(\epsilon\sqrt{i\log i} < |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \\
&\leqslant C \sum_{i\leqslant H(\epsilon)} E|X_1|^2 I\big(\epsilon\sqrt{i\log i} < |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \sum_{n=1}^i n^{-1}(\log n)^{-1+\delta} \\
&\leqslant C \sum_{i\leqslant H(\epsilon)} (\log i)^{-1} E|X_1|^2\big(\log|X_1|\big)^\delta I\big(\epsilon\sqrt{i\log i} < |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \\
&\leqslant C \sum_{i\leqslant H(\epsilon)} E|X_1|^2\big(\log|X_1|\big)^\delta I\big(\epsilon\sqrt{i\log i} < |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \\
&\leqslant C E|X_1|^2\big(\log|X_1|\big)^\delta < \infty. \qquad (2.15)
\end{aligned}$$

Combining (2.12)–(2.15), consequently, we obtain (2.11). □

Proposition 2.5. *Under the conditions of Theorem* 1.1,

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta} \sum_{n>H(\epsilon)} \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon\sqrt{n\log n}}^\infty 2x P\big(|N| \geqslant x/\sqrt{n}\big)\,dx = 0, \qquad (2.16)$$

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta} \sum_{n>H(\epsilon)} \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon\sqrt{n\log n}}^\infty 2x P\big(|S_n| \geqslant x\big)\,dx = 0. \qquad (2.17)$$

Proof. The proof of (2.16) is quite routine, we omit it. Applying Lemma 2.2, the proof of (2.17) are exposed as follows:

$$\begin{aligned}
&\sum_{n>H(\epsilon)} n^{-2}(\log n)^{\delta-1} \int_{\epsilon\sqrt{n\log n}}^\infty 2x P\big(|S_n| \geqslant x\big)\,dx \\
&\quad\leqslant C \sum_{n>H(\epsilon)} n^{-1}(\log n)^\delta \int_0^\infty 2(x+\epsilon) P\big(|S_n| \geqslant (x+\epsilon)\sqrt{n\log n}\big)\,dx \\
&\quad\leqslant C \sum_{n>H(\epsilon)} n^{-1}(\log n)^\delta \int_0^\infty 2(x+\epsilon) \Bigg\{ nP\big(|X_1| \geqslant 2(x+\epsilon)\sqrt{n\log n}\big)\,dx \\
&\qquad + \big((x+\epsilon)\sqrt{n\log n}\big)^{-q} n^{q/2} \exp\Bigg(\sum_{i=1}^{[\log n]} \rho\big(2^i\big)\Bigg)\,dx \\
&\qquad + \big((x+\epsilon)\sqrt{n\log n}\big)^{-q} n \exp\Bigg(\sum_{i=1}^{[\log n]} \rho^{2/q}\big(2^i\big)\Bigg) E|X_1|^q I\big(|X_1| \leqslant 2(x+\epsilon)\sqrt{n\log n}\big)\,dx \Bigg\}
\end{aligned}$$

$$=: C \sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon)(II_1+II_2+II_3)\,dx.$$

Recalling the moment condition, it suffices to prove that

$$\begin{aligned}
\sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon)II_1\,dx &\leqslant C \sum_{n>H(\epsilon)} (\log n)^{\delta} \int_{\epsilon}^{\infty} 2x P\big(|X_1| \geqslant 2x\sqrt{n\log n}\big)\,dx \\
&\leqslant C \sum_{n>H(\epsilon)} (\log n)^{\delta} E \int_{\epsilon}^{\infty} 2x I\big(|X_1| \geqslant 2x\sqrt{n\log n}\big)\,dx \\
&\leqslant C E \int_{\epsilon}^{\infty} \frac{X_1^2}{x} \big|\log|X_1| - \log x\big|^{\delta-1} I\big(|X_1| \geqslant x\big) I\big(|X_1| \geqslant \sqrt{M}\big)\,dx \\
&\leqslant C E X_1^2 \big|\log|X_1| - \log\epsilon\big|^{\delta} I\big(|X_1| \geqslant \sqrt{M}\big) \\
&\leqslant C E X_1^2 \big(\log|X_1|\big)^{\delta} + C E X_1^2 |\log\epsilon|^{\delta} < \infty.
\end{aligned} \tag{2.18}$$

In light of $q > 2\delta + 2$, the proof of part II_2 follows immediately. Consequently, turn to the last part II_3, it leads to

$$\begin{aligned}
&\sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon)II_3\,dx \\
&\leqslant \sum_{n>H(\epsilon)} n^{-q/2}(\log n)^{-q/2+\delta} E|X_1|^q \int_0^{\infty} (x+\epsilon)^{1-q} I\big(|X_1| \leqslant 2(x+\epsilon)\sqrt{n\log n}\big)\,dx \\
&= \sum_{n>H(\epsilon)} n^{-q/2}(\log n)^{-q/2+\delta} E|X_1|^q \int_0^{\infty} (x+\epsilon)^{1-q} I\big(|X_1| \leqslant \epsilon\sqrt{n\log n}\big)\,dx \\
&\quad + \sum_{n>H(\epsilon)} n^{-q/2}(\log n)^{-q/2+\delta} E|X_1|^q \int_0^{\infty} (x+\epsilon)^{1-q} I\big(\epsilon\sqrt{n\log n} < |X_1| \leqslant 2(x+\epsilon)\sqrt{n\log n}\big)\,dx \\
&=: II_4 + II_5.
\end{aligned}$$

A careful calculation shows that

$$\begin{aligned}
II_4 &\leqslant C\epsilon^{2-q} \sum_{n>H(\epsilon)} n^{-q/2}(\log n)^{-q/2+\delta} E|X_1|^q I\big(|X_1| \leqslant \epsilon\sqrt{n\log n}\big) \\
&\leqslant C\epsilon^{2-q} \sum_{n>H(\epsilon)} n^{-q/2}(\log n)^{-q/2+\delta} \sum_{i=1}^{n} E|X_1|^q I\big(\epsilon\sqrt{i\log i} \leqslant |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \\
&= C\epsilon^{2-q} \sum_{i>H(\epsilon)} E|X_1|^q I\big(\epsilon\sqrt{i\log i} \leqslant |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \sum_{n=i}^{\infty} n^{-q/2}(\log n)^{-q/2+\delta} \\
&\leqslant C\epsilon^{2-q} \sum_{i>H(\epsilon)} i^{-q/2+1}(\log i)^{-q/2+\delta} E|X_1|^q I\big(\epsilon\sqrt{i\log i} \leqslant |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \\
&\leqslant C \sum_{i>H(\epsilon)} (\log i)^{-1} E|X_1|^2 \big(\log|X_1|\big)^{\delta} I\big(\epsilon\sqrt{i\log i} \leqslant |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big) \\
&\leqslant C \sum_{i>H(\epsilon)} E|X_1|^2 \big(\log|X_1|\big)^{\delta} I\big(\epsilon\sqrt{i\log i} \leqslant |X_1| \leqslant \epsilon\sqrt{(i+1)\log(i+1)}\big)
\end{aligned}$$

$$\leqslant CE|X_1|^2\big(\log|X_1|\big)^\delta I\big(|X_1|\geqslant\epsilon\sqrt{H(\epsilon)\log H(\epsilon)}\big)<\infty. \tag{2.19}$$

Finally, we have

$$\begin{aligned}
II_5 &\leqslant C\sum_{n>H(\epsilon)} n^{-q/2}(\log n)^{-q/2+\delta}E|X_1|^q I\big(|X_1|>\epsilon\sqrt{n\log n}\big)\int_0^\infty (x+\epsilon)^{1-q} I\big(|X_1|\leqslant 2(x+\epsilon)\sqrt{n\log n}\big)\,dx\\
&\leqslant C\sum_{n>H(\epsilon)} n^{-1}(\log n)^{-1+\delta}\sum_{i=n}^\infty E|X_1|^2 I\big(\epsilon\sqrt{i\log i}<|X_1|\leqslant\epsilon\sqrt{(i+1)\log(i+1)}\big)\\
&\leqslant C\sum_{i>H(\epsilon)} E|X_1|^2 I\big(\epsilon\sqrt{i\log i}<|X_1|\leqslant\epsilon\sqrt{(i+1)\log(i+1)}\big)\sum_{n=1}^i n^{-1}(\log n)^{-1+\delta}\\
&\leqslant C\sum_{i>H(\epsilon)} (\log i)^{-1}E|X_1|^2\big(\log|X_1|\big)^\delta I\big(\epsilon\sqrt{i\log i}<|X_1|\leqslant\epsilon\sqrt{(i+1)\log(i+1)}\big)\\
&\leqslant CE|X_1|^2\big(\log|X_1|\big)^\delta I\big(|X_1|\geqslant\epsilon\sqrt{H(\epsilon)\log H(\epsilon)}\big)<\infty.
\end{aligned} \tag{2.20}$$

With the aid of (2.18)–(2.20), one can complete the proof of (2.17). □

Proof of Theorem 1.1. The proof follows immediately from the above five propositions. □

3. Proofs of Theorems 1.2 and 1.3

Note that (1.7) is the maximal version of (1.6), if we make some modification of the proof of (1.6), Theorem 1.2 will follow from Lemma 2.2 together with the following lemma.

Lemma 3.1. *(See Shao [18].) Let $\{X_n:\ n\geqslant 1\}$ be a strictly stationary sequences of ρ-mixing random variables satisfying $EX_1=0$, $EX_1^2<\infty$ and $\sum_{n=1}^\infty\rho(2^n)<\infty$. If $\sigma_n^2=ES_n^2\to\infty$, $n\to\infty$, then $\lim_{n\to\infty}\sigma_n^2/n=\sigma^2$. Furthermore, if $\sigma>0$, then $W_n\Rightarrow W$, where $W_n(t)=S_{[nt]}/\sigma\sqrt{n}$, $0\leqslant t\leqslant 1$, "$\Rightarrow$" means weak convergence in $D[0,1]$ with Skorohod topology, in particular,*

$$M_n/\sigma\sqrt{n}\xrightarrow{\mathscr{D}}\sup_{0\leqslant t\leqslant 1}\big|W(t)\big|. \tag{3.1}$$

Indeed, it suffices to show that

$$\begin{aligned}
\sum_{n=2}^\infty\frac{(\log n)^{\delta-1}}{n^2}EM_n^2 I\big(M_n\geqslant\epsilon\sqrt{n\log n}\big) &= \epsilon^2\sum_{n=2}^\infty\frac{(\log n)^\delta}{n}P\big(M_n\geqslant\epsilon\sqrt{n\log n}\big)\\
&\quad+\sum_{n=2}^\infty\frac{(\log n)^{\delta-1}}{n^2}\int_{\epsilon\sqrt{n\log n}}^\infty 2xP(M_n\geqslant x)\,dx\\
&=: I_3+I_4.
\end{aligned}$$

To pave the way for the proofs of I_3 and I_4, several propositions will be given as follows.

Proposition 3.1. *Suppose that $\{W(t):\ t\geqslant 0\}$ is a standard Wiener process* (*Brownian motion*)*. Then, for any $0<\delta\leqslant 1$,*

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta+2}\sum_{n=1}^\infty\frac{(\log n)^\delta}{n}P\Big(\sup_{0\leqslant s\leqslant 1}\big|W(s)\big|\geqslant\epsilon\sqrt{2\log n}\Big)=\frac{2E|N|^{2\delta+2}}{\delta+1}\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)^{2\delta+2}}. \tag{3.2}$$

Proof. Refer to Huang et al. [5]. □

Proposition 3.2. *For any* $0 < \delta \leqslant 1$, *we have*

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\Big(\sup_{0\leqslant t\leqslant 1}\big|W(t)\big|\geqslant x/\sqrt{n}\Big)\,dx=\frac{2E|N|^{2\delta+2}}{\delta(\delta+1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2\delta+2}}. \tag{3.3}$$

Proof. By Fubini's theorem together with Proposition 3.1, it turns out that

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}n^{-2}(\log n)^{\delta-1}\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\Big(\sup_{0\leqslant t\leqslant 1}\big|W(t)\big|\geqslant x/\sqrt{n}\Big)\,dx\\
&\quad=\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\int_{2}^{\infty}y^{-1}(\log y)^{\delta-1}dy\int_{\epsilon\sqrt{\log y}}^{\infty}2sP\Big(\sup_{0\leqslant t\leqslant 1}\big|W(t)\big|\geqslant s\Big)\,ds\\
&\quad=\int_{0}^{\infty}2u^{2\delta-1}\,du\int_{u}^{\infty}2sP\Big(\sup_{0\leqslant t\leqslant 1}\big|W(t)\big|\geqslant s\Big)\,ds\\
&\quad=\int_{0}^{\infty}2sP\Big(\sup_{0\leqslant t\leqslant 1}\big|W(t)\big|\geqslant s\Big)\,ds\int_{0}^{s}2u^{2\delta-1}\,du\\
&\quad=\frac{2}{\delta}\int_{0}^{\infty}s^{2\delta+1}P\Big(\sup_{0\leqslant t\leqslant 1}\big|W(t)\big|\geqslant s\Big)\,ds\\
&\quad=\frac{2E|N|^{2\delta+2}}{\delta(\delta+1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2\delta+2}}. \qquad\Box
\end{aligned} \tag{3.4}$$

Proof of Theorem 1.2. Along the same lines as that of the proof of Theorem 1.1, together with Lemma 3.1 and Propositions 3.1 and 3.2, one can easily complete the proof. □

To prove Theorem 1.3, the definition and the proposition are necessary, which reads as follows.

$$\psi(n):=\sup_{k\geqslant 1}\sup_{A\in\mathcal{F}_k^-}\sup_{B\in\mathcal{F}_{k+n}^+,\,P(A)P(B)>0}\frac{|P(AB)-P(A)P(B)|}{(P(A)P(B))^{1/2}},$$

the sequence $\{X_n\colon n\geqslant 1\}$ is said to be ψ-mixing, if $\psi(n)\to 0$ as $n\to\infty$. It is well known that $\psi(n)\leqslant\rho(n)$.

Proposition 3.3. *Suppose that* $\lim_{n\to\infty}ES_n^2/n=\sigma^2>0$, *and* $\sum_{n=1}^{\infty}\rho(n)<\infty$, *if* (1.9) *holds. Then, for any* $\epsilon>0$,

$$P\big(M_n\geqslant\epsilon\sqrt{n\log n}\,\big)\geqslant\lambda nP\big(|X_1|\geqslant 2\epsilon\sqrt{n\log n}\big), \tag{3.5}$$

where λ *is some positive constant.*

Proof. We first show that

$$nP\big(|X_1|\geqslant 2\epsilon\sqrt{n\log n}\big)\to 0\quad\text{as } n\to\infty. \tag{3.6}$$

According to (1.9), we have, for any $\epsilon>0$,

$$\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\big(M_n\geqslant\epsilon\sqrt{n\log n}\big)<\infty. \tag{3.7}$$

Observe that $|X_n|=|S_n-S_{n-1}|\leqslant|S_n|+|S_{n-1}|$. Then, from (3.7), we have

$$\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\Big(\max_{1\leqslant k\leqslant n}|X_k|\geqslant 2\epsilon\sqrt{n\log n}\Big)<\infty,\quad \epsilon>0,\tag{3.8}$$

and

$$\begin{aligned}&\big(\log 2^j\big)^{\delta}P\Big(\max_{1\leqslant k\leqslant 2^j}|X_k|\geqslant\epsilon\sqrt{2^j\log 2^j}\Big)\\&\quad\leqslant C\sum_{n=2^j}^{2^{j+1}}\frac{(\log n)^{\delta}}{n}P\Big(\max_{1\leqslant k\leqslant n}|X_k|\geqslant\epsilon\sqrt{n\log n/2}/2\Big)\to 0,\quad j\to\infty.\end{aligned}\tag{3.9}$$

Taking $a=(\log n)^{\delta}$, notice that $P(AB)=P(A)P(B)+\{P(A)P(\overline{B})-P(A\overline{B})\}$ and the relation between ρ-mixing and ψ-mixing. One can get

$$\begin{aligned}&P\Big(\max_{1\leqslant k\leqslant n}|X_k|\geqslant\epsilon\sqrt{n\log n}\Big)\\&\quad\geqslant P\left(\bigcup_{i=1}^{[n/a]}\big(|X_{ia}|\geqslant\epsilon\sqrt{n\log n}\big)\right)\\&\quad=\sum_{i=1}^{[n/a]}P\Big(\max_{j<i}|X_{ja}|<\epsilon\sqrt{n\log n},|X_{ia}|\geqslant\epsilon\sqrt{n\log n}\Big)\\&\quad\geqslant\sum_{i=1}^{[n/a]}\Big\{P\Big(\max_{j<i}|X_{ja}|<\epsilon\sqrt{n\log n}\Big)P\big(|X_{ia}|\geqslant\epsilon\sqrt{n\log n}\big)\\&\qquad-\rho(a)P^{1/2}\Big(\max_{j<i}|X_{ja}|\geqslant\epsilon\sqrt{n\log n}\Big)P^{1/2}\big(|X_{ia}|\geqslant\epsilon\sqrt{n\log n}\big)\Big\}\\&\quad\geqslant[n/a]P\big(|X_1|\geqslant\epsilon\sqrt{n\log n}\big)\Big\{P\Big(\max_{1\leqslant j<n}|X_j|<\epsilon\sqrt{n\log n}\Big)-\rho(a)[n/a]^{1/2}\Big\}.\end{aligned}\tag{3.10}$$

Using the fact that $P(\max_{1\leqslant j<2^i}|X_j|<\epsilon\sqrt{2^i\log 2^i})\to 1$ as $i\to\infty$ and $\sum_{n=1}^{\infty}\rho(n)<\infty$, which in turn implies (3.6). Finally, note that $\sum_{n=1}^{\infty}\rho(n)<\infty$, so one can choose a positive integer m such that $\sum_{i=1}^{\infty}\rho(mi)<1$. It follows that

$$\begin{aligned}&P\Big(\max_{1\leqslant k\leqslant n}|X_k|\geqslant\epsilon\sqrt{n\log n}\Big)\\&\quad\geqslant\sum_{i=1}^{[n/m]}P\big(|X_i|\geqslant\epsilon\sqrt{n\log n}\big)-\sum_{1\leqslant i<j\leqslant[n/m]}P\big(|X_{im}|\geqslant\epsilon\sqrt{n\log n},|X_{jm}|\geqslant\epsilon\sqrt{n\log n}\big)\\&\quad\geqslant[n/m]P\big(|X_1|\geqslant\epsilon\sqrt{n\log n}\big)-\big([n/m]P\big(|X_1|\geqslant\epsilon\sqrt{n\log n}\big)\big)^2-[n/m]P\big(|X_1|\geqslant\epsilon\sqrt{n\log n}\big)\sum_{i=1}^{[n/m]}\rho(mi)\\&\quad\geqslant[n/m]P\big(|X_1|>\epsilon\sqrt{n\log n}\big)\left(1-[n/m]P\big(|X_1|\geqslant\epsilon\sqrt{n\log n}\big)-\sum_{i=1}^{[n/m]}\rho(mi)\right).\end{aligned}\tag{3.11}$$

Taking $\lambda=[n/m]/n$, thus (3.5) follows from (3.6) and (3.11). □

Proof of Theorem 1.3. According to Theorem 1.2, the sufficient part is obvious. The necessary part follows from Propositions 3.3 together with the following result:

$$\infty>\sum_{n=2}^{\infty}n^{-1}(\log n)^{\delta}\int_0^{\infty}2(x+1)P\big(M_n\geqslant(x+1)\sqrt{n\log n}\big)\,dx$$

$$\geqslant C\sum_{n=N}^{\infty} n^{-1}(\log n)^{\delta}\int_0^{\infty}(x+1)nP\big(|X_1|\geqslant 2(x+1)\sqrt{n\log n}\big)\,dx$$

$$\geqslant CE\int_0^{\infty}(x+1)\sum_{n=N}^{\infty}(\log n)^{\delta} I\left(\sqrt{n\log n}\leqslant \frac{|X_1|}{2(x+1)}\right) I\left(x+1\leqslant \frac{|X_1|}{2}\right)dx$$

$$\geqslant CEX_1^2\big(\log|X_1|\big)^{\delta}. \qquad \square$$

Acknowledgment

Many thanks are due to the referee for a careful reading of the manuscript and for valuable suggestions.

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